Boundedness of the differentiation operator in model spaces and application to Peller type inequalities

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DIFFERENTIATION OPERATOR FROM MODEL SPACES TO BERGMAN SPACES AND PELLER TYPE INEQUALITIES

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Abstract. Given an inner function Θ in the unit disc $\mathbb{D}$, we study the boundedness of the differentiation operator which acts from the model subspace $K_\Theta = \left(\Theta H^2\right)^\perp$ of the Hardy space H^2, equiped with the $BMOA$-norm, to some radial-weighted Bergman space. As an application, we generalize Peller's inequality for Besov norms of rational functions f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}}$.

1. Introduction and notations

A well-known inequality by Vladimir Peller (see inequality (2.1) below) majorizes a Besov norm of any rational function f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$ in terms of its $BMOA$-norm and its degree n. The original proof of Peller is based on his description of Hankel operators in the Schatten classes. One of the aims of this paper is to give a short and direct proof of this inequality and extend it to more general radial-weighted Bergman norms. Our proof combines integral representation for the derivative of f (which come from the theory of model spaces) and the generalization of a theorem by E.M. Dyn'kin. The corresponding inequalities are obtained in terms of radial-weighted Bergman norms of the derivative of finite Blaschke products (of degree $n = \deg f$), instead of n itself. The finite Blaschke products in question have the same poles as f. The study of radial-weighted Bergman norms of the derivatives of finite Blaschke products of degree n and their asymptotic as n tends to $+\infty$ is of independent interest. A contribution to this topic, which we are going to exploit here, was given by J. Arazy, S.D. Fisher and J. Peetre.

Let $\mathcal{P}_n$ be the space of complex analytic polynomials of degree at most n and let

$$\mathcal{R}_n^+ = \left\{ \frac{P}{Q} \,:\, P, Q \in \mathcal{P}_n,\ Q(\xi) \neq 0 \text{ for } |\xi| \leq 1 \right\}$$

be the set of rational functions of degree at most n with poles outside of the closed unit disc $\overline{\mathbb{D}}$. In this paper, we consider the norm of a rational function $f \in \mathcal{R}_n^+$ in different spaces of analytic functions in the open unit disc $\mathbb{D} = \{\xi : |\xi| < 1\}$.

1.1. **Some Banach spaces of analytic functions.** We denote by $\mathcal{H}ol(\mathbb{D})$ the space of all holomorphic functions in $\mathbb{D}$.

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1.1.1. *The Besov spaces* B_p. A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Besov space B_p, $1 < p < \infty$, if and only if

$$\|f\|_{B_p} = |f(0)| + \|f\|^{\star}_{B_p} < +\infty,$$

where $\|f\|^{\star}_{B_p}$ is the seminorm defined by

$$\|f\|^{\star}_{B_p} = \left(\int_{\mathbb{D}} (1-|u|^2)^{p-2} \left| f'(u) \right|^p \mathrm{d}A(u) \right)^{\frac{1}{p}},$$

A being the normalized area measure on $\mathbb{D}$.

For the case $0 < p \leq 1$ the definition of the Besov norm requires a modification:

$$\|f\|_{B_p} = \sum_{j=0}^{k-1} |f^{(j)}(0)| + \|f\|^{\star}_{B_p}, \qquad \|f\|^{\star}_{B_p} = \left(\int_{\mathbb{D}} \left| f^{(k)}(u) \right|^p (1-|u|^2)^{pk-2} \mathrm{d}A(u) \right)^{\frac{1}{p}},$$

where k is the smallest positive integer such that $pk > 1$. We refer to [Pee, Tri, BeLo] for general properties of Besov spaces.

A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space B_∞ (known as the Bloch space) if and only if $\|f\|_{B_\infty} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1-|z|) < \infty$.

1.1.2. *The radial-weighted Bergman spaces* $A_p(w)$. The radial-weighted Bergman space $A_p(w)$, $1 \leq p < \infty$, is defined as:

$$A_p(w) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|^p_{A_p(w)} = \int_{\mathbb{D}} w(|u|) \left| f(u) \right|^p \mathrm{d}A(u) < \infty \right\},$$

where the weight w satisfies $w \geq 0$ and $\int_0^1 w(r)\, \mathrm{d}r < \infty$. The classical power weights $w(r) = w_\alpha(r) = (1-r^2)^\alpha$, $\alpha > -1$, are of special interest; in this case we put $A_p(\alpha) = A_p(w_\alpha)$. We refer to [HKZ] for general properties of weighted Bergman spaces.

1.1.3. *The spaces* $A^1_p(\alpha)$. A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space $A^1_p(\alpha)$, $1 \leq p \leq +\infty$, $\alpha > -1$, if and only if

$$\|f\|_{A^1_p(\alpha)} = |f(0)| + \|f'\|_{A_p(\alpha)} < +\infty.$$

We also define the $A^1_p(\alpha)$-seminorm by $\|f\|^{\star}_{A^1_p(\alpha)} = \|f'\|_{A_p(\alpha)}$. Note that the spaces B_p and $A^1_p(p-2)$ coincide for $1 < p < +\infty$.

1.1.4. *The space BMOA*. There are many ways to define $BMOA$; see [Gar, Chapter 6]. For the purposes of this paper we choose the following one: a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the $BMOA$ space (of analytic functions of bounded mean oscillation) if and only if

$$\|f\|_{BMOA} = \inf \|g\|_{L^\infty(\mathbb{T})} < +\infty,$$

where the infimum is taken over all $g \in L^\infty(\mathbb{T})$, $\mathbb{T} = \{\xi : |\xi| = 1\}$ being the unit circle, for which the representation

$$f(\xi) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(u)}{u - \xi} \mathrm{d}u, \qquad |\xi| < 1,$$

holds. Recall that $BMOA$ is the dual space of the Hardy space H^1 under the pairing

$$\langle f,\, g\rangle = \int_{\mathbb{T}} f(u)\overline{g(u)}\mathrm{d}u, \qquad f \in H^1,\ g \in BMOA,$$

where this integral must be understood as the extension of the pairing acting on a dense subclass of H^1, see [Bae, p. 23].

1.2. **Model spaces.**

1.2.1. *General inner functions.* By H^p, $1 \le p \le \infty$, we denote the standard Hardy spaces (see [Gar, Nik]). Recall that H^2 is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(w) = \frac{1}{1-\overline{\lambda}w}, \qquad \lambda,\, w \in \mathbb{D},$$

known as the Szegö kernel (or the Cauchy kernel) associated with λ. Thus $\langle f,\, k_\lambda\rangle = f(\lambda)$ for all $f \in H^2$ and for all $\lambda \in \mathbb{D}$, where $\langle \cdot,\, \cdot\rangle$ is the scalar product on H^2.

Let Θ be an *inner function*, i.e., $\Theta \in H^\infty$ and $|\Theta(\xi)| = 1$ a.e. $\xi \in \mathbb{T}$. We define the model subspace K_Θ of the Hardy space H^2 by

$$K_\Theta = \left(\Theta H^2\right)^\perp = H^2 \ominus \Theta H^2.$$

By the famous theorem of Beurling, these and only these subspaces of H^2 are invariant with respect to the backward shift operator. We refer to [Nik] for the general theory of the spaces K_Θ and their numerous applications.

For any *inner function* Θ, the reproducing kernel of the model space K_Θ corresponding to a point $\xi \in \mathbb{D}$ is of the form

$$k_\lambda^\Theta(w) = \frac{1-\overline{\Theta(\lambda)}\Theta(w)}{1-\overline{\lambda}w}, \qquad \lambda,\, w \in \mathbb{D},$$

that is $\left\langle f,\, k_\lambda^\Theta\right\rangle = f(\lambda)$ for all $f \in K_\Theta$ and for all $\lambda \in \mathbb{D}$.

1.2.2. *The case of finite Blaschke products.* From now on, for any $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^{n} b_{\lambda_k},$$

where $b_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda}z}$, is the elementary Blaschke factor corresponding to $\lambda \in \mathbb{D}$. It is well known that if

$$\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\} \in \mathbb{D}^n,$$

where every λ_s is repeated according to its multiplicity n_s, $\sum_{s=1}^t n_s = n$, then

$$K_{B_\sigma} = H^2 \ominus B_\sigma H^2 = \overline{\operatorname{span}}\{k_{\lambda_j,\, i} :\ 1 \le j \le t,\, 0 \le i \le n_j - 1\},$$

where for $\lambda \ne 0$, $k_{\lambda,\, i} = \left(\frac{d}{d\overline{\lambda}}\right)^i k_\lambda$ and $k_\lambda = \frac{1}{1-\overline{\lambda}z}$ is the standard Cauchy kernel at the point λ, whereas $k_{0,\, i} = z^i$. Thus the subspace K_{B_σ} consists of rational functions of the form p/q, where $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{P}_n$, with the poles $1/\overline{\lambda}_1, \dots, 1/\overline{\lambda}_n$ of corresponding multiplicities (including possible poles at ∞). Hence, if $f \in \mathcal{R}_n^+$ and $1/\overline{\lambda}_1, \dots, 1/\overline{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \dots, \lambda_n)$.

From now on, for two positive functions a and b, we say that a is dominated by b, denoted by $a \lesssim b$, if there is a constant $C > 0$ such that $a \leq Cb$; we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

2. Main results

2.1. **Main ingredients.** In 1980 V. Peller proved in his seminal paper [Pel1] that

$$\|f\|_{B_p} \leq c_p n^{\frac{1}{p}} \|f\|_{BMOA} \tag{2.1}$$

for any $f \in \mathcal{R}_n^+$ and $1 \leq p \leq +\infty$, where c_p is a constant depending only on p. Later, this result was extended to the range $p > 0$ independently and with different proofs by Peller [Pel2], S. Semmes [Sem] and also by A. Pekarskii [Pek1] who found a proof which does not use the theory of Hankel operators (see also [Pek2]).

The aim of the present article is:

(1) study the boundedness of the differention operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$, $1 < p < +\infty$, $\alpha > -1$, and
(2) generalize Peller's result (2.1) replacing the B_p-seminorm by the $A_p^1(\alpha)$-one.

In both of these problems, we make use of a method based on two main ingredients:

- integral representation for the derivative of functions in K_Θ or in $\mathcal{R}_n^+$, and
- a generalization of a theorem by E.M. Dyn'kin, see Subsection 2.2.3.

One more tool (that we will need in problem (2)) is the estimate of B_p-seminorms of finite Blaschke products by Arazy, Fischer and Peetre [AFP].

2.2. **Main results.** Let us consider the differentation operator $Df = f'$ and the shift and the backward shift operators defined respectively by

$$Sf = zf, \qquad S^\star f = \frac{f - f(0)}{z}, \tag{2.2}$$

for any $f \in \mathcal{H}ol(\mathbb{D})$. From now on, for any inner function Θ, we put

$$\widetilde{\Theta} = z\Theta = S\Theta.$$

2.2.1. *Boundedness of the differentiation operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$.* Let us first discuss the boundedness of the operator D from $BMOA$ to $A_p(\alpha)$. The following (essentially well-known) proposition gives necessary and sufficient conditions on p and α so that a continuous embedding $BMOA \subset A_p^1(\alpha)$ hold.

Proposition 2.1. *Let $\alpha > -1$ and $1 \leq p < \infty$. Then $BMOA \subset A_p^1(\alpha)$ if and only if either $\alpha > p - 1$ or $\alpha = p - 1$ and $p \geq 2$.*

Now, we consider an arbitrary *inner function* Θ. Our first main result gives necessary and sufficient conditions under which the differentiation operator

$$D: \ (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$$

is bounded. When this is the case, we estimate its norm in terms of $\|\Theta'\|_{A_p(\alpha)}$.

Theorem 2.2. *Let $1 < p < \infty$ and $\alpha > -1$. Then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$ is bounded if and only if $\Theta' \in A_p(\alpha)$.*

Moreover, one can distinguish three cases:

(a) If $\alpha > p-1$ or $\alpha = p-1$, and $p \geq 2$ then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$ is bounded.

(b) If $p-2 < \alpha < p-1$, then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$ is bounded if and only if $\Theta' \in A_1(\alpha - p + 1)$.

(c) If $\alpha \leq p-2$, then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$ is bounded if and only if Θ is a finite Blaschke product.

In cases (b) and (c), we have

$$\|D\| \lesssim \|\Theta'\|_{A_p(\alpha)} \lesssim \|D\| + \text{const}, \tag{2.3}$$

with constants depending on p and α only.

Remark. 1. In the cases (*b*) and (*c*), to show that their conditions are equivalent to the inclusion $\Theta' \in A_p(\alpha)$ we use a theorem by P.R. Ahern [Ahe1] and its generalizations by I.E. Verbitsky [Ver] and A. Gluchoff [Glu]. We do not know whether the inclusions $\Theta' \in A_p(\alpha)$ and $\Theta' \in A_1(\alpha - p + 1)$ are equivalent for $\alpha = p - 1$ and $1 < p < 2$.

2. The membership of Blaschke products in various function spaces is a well-studied topic. Besides the above-cited papers by Ahern, Gluchoff and Verbitsky, let us mention the papers by Ahern and D.N. Clark [AC1, AC2] and recent works by D. Girela, J. Peláez, D. Vukotić, and A. Aleman [GPV, AV].

2.2.2. *Generalization of Peller's inequalities.* In the following theorem, we give a generalization of Peller's inequality (2.1).

Theorem 2.3. *Let $f \in \mathcal{R}_n^+$, $\deg f = n$ and $\sigma \in \mathbb{D}^n$ be the set of its poles counting multiplicities (including poles at ∞). For any $\alpha > -1$, $1 < p < \infty$, and $p > 1 + \alpha$, we have*

$$\|f\|^\star_{A_p^1(\alpha)} \leq K_{p,\alpha}\, \|f\|_{BMOA}\, \|B'_\sigma\|_{A_p(\alpha)}, \tag{2.4}$$

where $K^p_{p,\alpha} = \frac{2^{\frac{1}{\alpha+1}}}{2^{\frac{1}{\alpha+1}}-1}\left(\frac{p}{p-1-\alpha}\right)^p 2^{p+1}$.

Remark. The inequality (2.4) is sharp up to a constant in the following sense: for $f = B_\sigma$ we, obviously, have $\|f\|^\star_{A_p^1(\alpha)} = \|f\|_{BMOA}\|B'_\sigma\|_{A_p(\alpha)}$ (note that $\|B_\sigma\|_{BMOA} = 1$).

Let us show how Peller's inequality (2.1) for $1 < p < \infty$ follows from Theorem 2.3. For $\alpha = p - 2$, we have

$$\|f'\|_{A_p(\alpha)} = \|f\|^\star_{B_p}, \qquad \|B'_\sigma\|_{A_p(\alpha)} = \|B_\sigma\|^\star_{B_p}.$$

To deduce Peller's inequalities it remains to apply the following theorem by Arazy, Fischer and Peetre [AFP]: if $1 \leq p \leq \infty$, then there exist absolute positive constants m_p and M_p such that

$$m_p n^{\frac{1}{p}} \leq \|B\|^\star_{B_p} \leq M_p n^{\frac{1}{p}}. \tag{2.5}$$

for any Blaschke product of degree n. Then we obtain for $1 < p < \infty$,

$$\|f\|_{B_p}^{\star} \leq K_{p,\,p-2}^{\frac{1}{p}} M_p \|f\|_{BMOA} (n+1)^{\frac{1}{p}} \lesssim n^{\frac{1}{p}} \|f\|_{BMOA}\,.$$

To make the expositions self-contained, we give in Section 5 a very simple proof of the upper estimate in (2.5) (which is slightly different from the proof by D. Marshall presented in [AFP]).

The method of integral repesentations for higher order derivatives in model spaces allows to prove Peller's inequalities also for $0 < p \leq 1$. In Section 6 we present the proof for the case $p > \frac{1}{2}$.

2.2.3. *Generalization of a theorem by Dyn'kin.* E.M. Dyn'kin proved in [Dyn, Theorem 3.2] that

$$\int_{\mathbb{D}} \left(\frac{1-|B(u)|^2}{1-|u|^2} \right)^2 \mathrm{d}A(u) \leq 8(n+1), \tag{2.6}$$

for any finite Blaschke product B of degree n.

From now on, for any *inner function* Θ and for any $\alpha > -1$, $p > 1$, we put

$$I_{p,\,\alpha}(\Theta) = \int_{\mathbb{D}} (1-|u|^2)^{\alpha} \left(\frac{1-|\Theta(u)|^2}{1-|u|^2} \right)^p \mathrm{d}A(u). \tag{2.7}$$

Dyn'kin's Theorem can be stated as follows: *for any finite Blaschke product B of degree n, we have*

$$I_{2,\,0}(B) \leq 8(n+1).$$

Here, we generalize this result to the case $\alpha > -1$, $p > 1$ and $p > 1 + \alpha$. This generalization is the key step of the proof of Theorem 2.3.

Theorem 2.4. *Let $1 < p < \infty$, $\alpha > -1$ and $p > 1 + \alpha$. Then,*

$$\|\Theta'\|_{A_p(\alpha)}^p \leq I_{p,\,\alpha}(\Theta) \leq K_{p,\,\alpha} \|\Theta'\|_{A_p(\alpha)}^p\,,$$

where $K_{p,\,\alpha}$ is the same constant as in Theorem 2.3.

The paper is organized as follows. We first focus in Section 3 on the generalization of Dyn'kin's result. In Section 4, Proposition 2.1 and Theorem 2.2 are proved, while Section 5 is devoted to the proof of Peller type inequalities (Theorem 2.3). The case $\frac{1}{2} < p \leq 1$ in Peller's inequality is considered in Section 6. In Section 7, we discuss some estimates of radial-weighted Bergman norms of Blaschke products. Finally, in Section 8 we discuss some related inequalities by Dolzhenko for which we give a very simple proof for the case $1 \leq p \leq 2$ based on Dyn'kin's estimate and suggest a way to extend these inequalities to the range $p > 2$.

3. Generalization of Dyn'kin's Theorem

The aim of this Section is to prove Theorem 2.4. The lower bound follows trivially from the Schwarz–Pick inequality applied to Θ. The main ideas for the proof of the upper bound come from [Dyn, Theorem 3.2]. In this Section, Θ is an arbitrary inner function.

Lemma 3.1. *For $p>1$, $\alpha>-1$ and $p>1+\alpha$, we have*

$$I_{p,\alpha}(\Theta)\leq 2^p\int_0^{2\pi}\int_0^1(1-r)^\alpha\left(\frac{1}{1-r}\int_r^1|\Theta'(se^{i\theta})|\mathrm{d}s\right)^p\mathrm{d}r\frac{\mathrm{d}\theta}{\pi}.$$

Proof. Writing the integral $I_{p,\alpha}(\Theta)$ in polar coordinates, and using the fact that

$$1-|\Theta(u)|^2\leq 2(1-|\Theta(u)|),$$

we obtain

$$\begin{aligned}
I_{p,\alpha}(\Theta) &\leq 2^p\int_0^1 r(1-r^2)^{\alpha-p}\left(\int_0^{2\pi}(1-|\Theta(re^{i\theta})|)^p\frac{\mathrm{d}\theta}{\pi}\right)\mathrm{d}r\\
&\leq 2^p\int_0^1 r(1-r)^{\alpha-p}\left(\int_0^{2\pi}|\Theta(e^{i\theta})-\Theta(re^{i\theta})|^p\frac{\mathrm{d}\theta}{\pi}\right)\mathrm{d}r\\
&\leq 2^p\int_0^1 r(1-r)^{\alpha-p}\left(\int_0^{2\pi}\left(\int_r^1|\Theta'(se^{i\theta})|\mathrm{d}s\right)^p\frac{\mathrm{d}\theta}{\pi}\right)\mathrm{d}r\\
&\leq 2^p\int_0^{2\pi}\int_0^1(1-r)^\alpha\frac{1}{(1-r)^p}\left(\int_r^1|\Theta'(se^{i\theta})|\mathrm{d}s\right)^p\mathrm{d}r\frac{\mathrm{d}\theta}{\pi},
\end{aligned}$$

which completes the proof of the lemma. □

We recall now a general version of the Hardy inequality, see [HLP, page 245], which after change of variables gives (as in [Ahe2, Lemma 7]):

Lemma 3.2. *If $h:(0,1)\to[0,+\infty)$, $p>1$, $\alpha>-1$ and $p>1+\alpha$, then*

$$\int_0^1(1-r)^\alpha\left(\frac{1}{1-r}\int_r^1 h(s)\mathrm{d}s\right)^p\mathrm{d}r\leq\left(\frac{p}{p-1-\alpha}\right)^p\int_0^1(1-r)^\alpha h(r)^p\mathrm{d}r.$$

Corollary 3.3. *Let $p>1$, $\alpha>-1$ and $p>1+\alpha$. Then,*

$$I_{p,\alpha}(\Theta)\leq C_{p,\alpha}\int_0^{2\pi}\int_0^1(1-r)^\alpha|\Theta'(re^{i\theta})|^p\mathrm{d}r\frac{\mathrm{d}\theta}{\pi},$$

where $C_{p,\alpha}=\left(\frac{p}{p-1-\alpha}\right)^p2^p$.

Proof. Combining estimates in Lemma 3.1 and Lemma 3.2 (setting $h(s)=h_\theta(s)=|\Theta'(se^{i\theta})|$, for any fixed $\theta\in[0,2\pi)$), we obtain

$$\int_0^1(1-r)^\alpha\left(\frac{1}{1-r}\int_r^1|\Theta'(se^{i\theta})|\mathrm{d}s\right)^p\mathrm{d}r\leq\left(\frac{p}{p-1-\alpha}\right)^p\int_0^1(1-r)^\alpha|\Theta'(re^{i\theta})|^p\mathrm{d}r.$$

Thus,

$$I_{p,\alpha}(\Theta)\leq\left(\frac{p}{p-1-\alpha}\right)^p2^p\int_0^{2\pi}\int_0^1(1-r)^\alpha|\Theta'(re^{i\theta})|^p\mathrm{d}r\frac{\mathrm{d}\theta}{\pi}$$

which completes the proof. □

Lemma 3.4. *Let any nonzero weight w satisfying $w \geq 0$ and $\int_0^1 w(r)\,\mathrm{d}r < \infty$. Let $\beta = \beta_w \in (0,1)$ such that $\int_0^1 w(r)\,dr = 2\int_0^\beta w(r)\,dr$. Then, for $f \in A_p(w)$, $1 \leq p < \infty$,*

$$\|f\|^p_{A_p(w)} \leq \int_0^{2\pi} w(r) \int_0^1 |f(re^{i\theta})|^p \mathrm{d}r \frac{\mathrm{d}\theta}{\pi}$$
$$\leq \frac{2}{\beta}\int_\beta^1 rw(r)\left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{\mathrm{d}\theta}{\pi}\right)\mathrm{d}r \leq \frac{2}{\beta}\|f\|^p_{A_p(w)}.$$

Proof. The proof follows easily from the fact that for any f in $\mathcal{H}ol\,(\mathbb{D})$, the function

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})|^p \frac{\mathrm{d}\theta}{\pi},$$

is nondecreasing on $[0,1]$. □

We are now ready to prove Theorem 2.4.

Proof. We first prove (2.4). Applying Lemma 3.4 with $f = \Theta'$ and $w(r) = (1-r^2)^\alpha$, $\alpha > -1$, and Corollary 3.3 we obtain that

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi}\int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p \mathrm{d}r \frac{\mathrm{d}\theta}{\pi} \leq \frac{2}{\beta} C_{p,\alpha} \|\Theta'\|^p_{A_p(\alpha)},$$

where $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha}\right)^p 2^p$, and $\beta = \beta_\alpha$ satisfies the condition

$$\int_\beta^1 w(r)\,\mathrm{d}r = \int_0^\beta w(r)\,\mathrm{d}r.$$

By a direct computation, we see that $\beta = \beta_\alpha$ is given by the equation $\frac{1-(1-\beta)^{\alpha+1}}{1+\alpha} = \frac{(1-\beta)^{\alpha+1}}{1+\alpha}$, which is equivalent to

$$\beta = \beta_\alpha = 1 - \frac{1}{2^{\frac{1}{\alpha+1}}}. \tag{3.1}$$

□

4. Proof of Proposition 2.1 and Theorem 2.2

4.1. **Proof of Proposition 2.1.** The statement for $\alpha > p-1$ is trivial. Indeed, by the standard Cauchy formula,

$$f'(u) = \left\langle f, \frac{z}{(1-\overline{u}z)^2}\right\rangle, \qquad u \in \mathbb{D},$$

and thus, bounding $|f'(u)|$ from above by $\|f\|_{BMOA}\left\|\frac{z}{(1-\bar{u}z)^2}\right\|_{H^1} = (1-|u|^2)^{-1}\|f\|_{BMOA}$, we get

$$\|f'\|^p_{A_p(\alpha)} \lesssim \|f\|^p_{BMOA}\int_{\mathbb{D}} (1-|u|)^{\alpha-p}\mathrm{d}A(u) \lesssim \|f\|^p_{BMOA}.$$

For $p \geq 2$ and $\alpha = p - 1$ we have

$$\begin{aligned} \|f'\|^p_{A_p(\alpha)} &= \int_{\mathbb{D}} (1 - |u|^2)\, |f'(u)|^p \, \mathrm{d}A(u) \\ &\leq \|f\|^{p-2}_{B_\infty} \int_{\mathbb{D}} (1 - |u|^2)\, |f'(u)|^2 \, \mathrm{d}A(u), \end{aligned}$$

where $\|f\|_{B_\infty}$ is the norm of f in the Bloch space. Since $\int_{\mathbb{D}} (1-|u|^2)\, |f'(u)|^2 \, \mathrm{d}A(u) \leq \|f\|^2_{H^2}$, $\|f\|_{H^2} \lesssim \|f\|_{BMOA}$ and $\|f\|_{B_\infty} \lesssim \|f\|_{BMOA}$, we conclude that

$$\|f'\|_{A_p(\alpha)} \lesssim \|f\|_{BMOA} . \tag{4.1}$$

Now we turn to the necessity of the restrictions on p and α for the estimate (4.1). If $\alpha < p - 1$ then it is well known that there exist interpolating Blaschke products B such that $B' \notin A_p(\alpha)$ (see, e.g., [Glu, Theorem 6], where an explicit criterion for the inclusion is given in terms of the zeros of B). Finally, by a result of S.A. Vinogradov [Vin, Lemma 1.6], if $f \in A^1_p(p-1)$, $1 \leq p < 2$, then, $\sum_{n=0}^\infty |\hat{f}(2^n)|^p < \infty$ (where $\hat{f}(n)$ stands for the n^{th} Taylor coefficient of f). Hence, $A^1_p(p-1)$ does not contain even some functions from the disc algebra, and so $BMOA \not\subseteq A^1_p(p-1)$ when $1 \leq p < 2$. □

4.2. **Integral representation for the derivative of functions in K_Θ.** An important ingredient of our proof is the following simple and well-known integral representation for the derivative of a function from a model space.

Lemma 4.1. *Let Θ be an inner function, $f \in K_\Theta$, $n \in \mathbb{N}$. We have*

$$f^{(n)}(u) = \left\langle f,\, z^n \left(k^\Theta_u\right)^{n+1} \right\rangle,$$

for any $u \in \mathbb{D}$.

Proof. For a fixed $u \in \mathbb{D}$, we have

$$f^{(n)}(u) = \left\langle f,\, \frac{z^n}{(1-\overline{u}z)^{n+1}} \right\rangle = \left\langle f,\, z^n \left(k^\Theta_u\right)^{n+1} \right\rangle.$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that $z^n(1-\overline{u}z)^{-n-1} - z^n \left(k^\Theta_u(z)\right)^{n+1} \in \Theta H^2$ and $f \perp \Theta H^2$. □

4.3. **Proof of the left-hand side inequality in (2.3).** Sufficiency of the condition $\Theta' \in A_p(\alpha)$ in Theorem 2.2 and the left-hand side inequality in (2.3) follow immediately from Theorem 2.4 and the following proposition.

Proposition 4.2. *Let $\alpha > -1$ and $1 < p < \infty$, let Θ be an inner function and $f \in K_\Theta$. Then we have*

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} \left(I_{p,\alpha}(\Theta)\right)^{\frac{1}{p}} .$$

Proof. We use the integral representation of f' from Lemma 4.1:

$$f'(u) = \left\langle f,\, z\left(k^\Theta_u\right)^2 \right\rangle = \int_{\mathbb{T}} f(\tau)\overline{\tau\left(k^\Theta_u(\tau)\right)^2} \mathrm{d}m(\tau),$$

for any $u \in \mathbb{D}$, and thus

$$\begin{aligned}
\|f'\|^p_{A_p(\alpha)} &= \int_{\mathbb{D}} (1-|u|^2)^\alpha \left| \int_{\mathbb{T}} f(\tau)\overline{\tau\,(k_u^\Theta(\tau))^2} \mathrm{d}m(\tau) \right|^p \mathrm{d}A(u) \\
&\leq \|f\|^p_{BMOA} \int_{\mathbb{D}} (1-|u|^2)^\alpha \left(\int_{\mathbb{T}} \left|k_u^\Theta(\tau)\right|^2 \mathrm{d}m(\tau) \right)^p \mathrm{d}A(u) \\
&= \|f\|^p_{BMOA} \int_{\mathbb{D}} (1-|u|^2)^\alpha \left(\frac{1-|\Theta(u)|^2}{1-|u|^2} \right)^p \mathrm{d}A(u),
\end{aligned}$$

which completes the proof. $\square$

It remains to combine Proposition 4.2 with Theorem 2.4 to complete the proof of the left-hand side inequality in (2.3).

4.4. **Proof of the right-hand side inequality in (2.3).** To prove the necessity of the inclusion $\Theta' \in A_p(\alpha)$ and the left-hand side inequality in (2.3), consider the test function

$$f = S^\star \Theta = \frac{\Theta - \Theta(0)}{z},$$

where $S^\star$ is the backward shift operator (2.2). It is well-known that f belongs to K_Θ and easy to check that $\|f\|_{BMOA} \leq 2$, whence

$$\|D\|_{\left(K_\Theta,\, \|\cdot\|_{BMOA}\right) \to A_p(\alpha)} \geq \frac{\|f'\|_{A_p(\alpha)}}{2}. \tag{4.2}$$

Now,

$$\|f'\|^p_{A_p(\alpha)} \geq \int_{\beta_\alpha}^1 r\, w(r) \int_{\mathbb{T}} |f'(r\xi)|^p \,\mathrm{d}m(\xi)\,\mathrm{d}r,$$

where β_α is given by (3.1) and thus,

$$\begin{aligned}
\|f'\|_{A_p(\alpha)} \geq &\left(\int_{\beta_\alpha}^1 r\, w(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p \mathrm{d}m(\xi)\,\mathrm{d}r \right)^{\frac{1}{p}} \\
&- \left(\int_{\beta_\alpha}^1 r\, w(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi)-\Theta(0)}{r^2\xi^2} \right|^p \mathrm{d}m(\xi)\,\mathrm{d}r \right)^{\frac{1}{p}}.
\end{aligned}$$

On one hand, applying Lemma 3.4 with $w = w_\alpha$ and $\beta = \beta_\alpha$ we obtain

$$\begin{aligned}
\int_{\beta_\alpha}^1 r\, w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p \mathrm{d}m(\xi)\,\mathrm{d}r &\geq \int_0^{2\pi} \int_{\beta_\alpha}^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p \mathrm{d}r \frac{\mathrm{d}\theta}{\pi} \\
&\geq \frac{\beta_\alpha}{2} \int_0^{2\pi} \int_0^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p \mathrm{d}r \frac{\mathrm{d}\theta}{\pi} = \frac{\beta_\alpha}{2} \parallel \Theta' \parallel^p_{A_p(\alpha)}.
\end{aligned}$$

On the other hand, since $\|f\|_{H^\infty} \leq 2$, we have

$$\int_{\beta_\alpha}^1 r\, w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi)-\Theta(0)}{r^2\xi^2} \right|^p \mathrm{d}m(\xi)\,\mathrm{d}r \leq 2^p \int_{\beta_\alpha}^1 \frac{w_\alpha(r)}{r^{p-1}} \mathrm{d}r \leq \frac{2^p}{\beta_\alpha^{p-1}} \int_{\beta_\alpha}^1 w_\alpha(r) \mathrm{d}r.$$

Finally, we conclude that

$$\|f'\|_{A_p(\alpha)} \geq \left(\frac{\beta_\alpha}{2}\right)^{\frac{1}{p}} \|\Theta'\|_{A_p(\alpha)} - 2\beta_\alpha^{\frac{1}{p}-1}\left(\int_{\beta_\alpha}^1 w_\alpha(r)\mathrm{d}r\right)^{\frac{1}{p}},$$

which, combined with (4.2), gives us the right-hand side inequality in (2.3). □

4.5. **Proof of Theorem 2.2.** To complete the proof of Theorem 2.2, we need to recall the following theorem proved by Ahern [Ahe1] for the case $1 \leq p \leq 2$ and generalized by Verbitsky [Ver] and Gluchoff [Glu] to the range $1 \leq p < \infty$. This theorem characterizes inner functions Θ whose derivative belong to $A_p(\alpha)$.

Theorem. ([Glu]) *Let Θ be an inner function, $1 \leq p < \infty$, and $\alpha > -1$.*
(i) If $\alpha > p-1$, then $\Theta' \in A_p(\alpha)$.
(ii) If $p-2 < \alpha < p-1$, then $\Theta' \in A_p(\alpha)$ if and only if $\Theta' \in A_1(\alpha - p + 1)$.
(iii) If $\alpha < p-2$ and $p > 1$, then $\Theta' \in A_p(\alpha)$ if and only if Θ is a finite Blaschke product.

Proof of Theorem 2.2. Statement (a) is contained in Proposition 2.1. In order to prove (b) and (c) of Theorem 2.2, we first remark that for $\alpha < p-1$, it follows from (2.3) that $D: (K_\Theta, \|\cdot\|_{BMOA}) \to A_p(\alpha)$ is bounded if and only if $\Theta' \in A_p(\alpha)$. A direct application of the above Ahern–Verbitsky–Gluchoff theorem completes the proof for $\alpha > p-2$. The case $\alpha = p-2$ follows from the Arazy–Fisher–Peetre inequality (2.5). □

5. Proof of Peller type inequalities

In this section we prove Theorem 2.3. From now on the inner function Θ is a finite Blaschke product. Recall that if $f \in \mathcal{R}_n^+$ and $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \ldots, \lambda_n)$.

We start with the proof of the upper bound in the Arazy–Fisher–Peetre inequality (2.5).

Lemma 5.1. *Let B be a finite Blaschke product with the zeros $\{z_j\}_{j=1}^n$. Then*

$$|B''(u)| \leq \sum_{j=1}^n \frac{1-|z_j|^2}{|1-\overline{z_j}u|^3} + \left(\frac{1-|B(u)|}{1-|u|}\right)^2, \qquad u \in \mathbb{D}.$$

Proof. Let $B = \prod_{j=1}^n b_{z_j}$, where $b_\lambda = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{1-\overline{\lambda}z}$. Then it is easy to see that

$$|B''(u)| \leq \sum_{j=1}^n \frac{1-|z_j|^2}{|1-\overline{z_j}u|^3}\left|\frac{B(u)}{b_{z_j}(u)}\right| + 2\sum_{1\leq j<k\leq n} \frac{1-|z_j|^2}{|1-\overline{z_j}u|^2}\frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}\left|\frac{B(u)}{b_{z_j}(u)b_{z_k}(u)}\right|, \tag{5.1}$$

To estimate the second sum in (5.1), first we note that $\frac{1-|\lambda|^2}{2|1-\overline{\lambda}u|^2} \leq \frac{1-|b_\lambda(u)|}{1-|u|} \leq \frac{2(1-|\lambda|^2)}{|1-\overline{\lambda}u|^2}$. Let us introduce the notations $B_j = \prod_{l=1}^{j-1} b_{z_l}$ (assuming $B_1 \equiv 1$) and $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$. Then

$$\frac{1-|B(u)|}{1-|u|} \asymp \sum_{j=1}^n |B_j(u)|\frac{1-|z_j|^2}{|1-\overline{z_j}u|^2} \asymp \sum_{k=1}^n |\widehat{B}_{k+1}(u)|\frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}. \tag{5.2}$$

It follows from the estimate $|B/(b_{z_j}b_{z_k})| \le |B_j\widehat{B}_{k+1}|$ and (5.2) that

$$\sum_{1\le j<k\le n}\frac{1-|z_j|^2}{|1-\overline{z_j}u|^2}\frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}\left|\frac{B(u)}{b_{z_j}(u)b_{z_k}(u)}\right| \le \left(\sum_{j=1}^n |B_j(u)|\frac{1-|z_j|^2}{|1-\overline{z_j}u|^2}\right)\times$$
$$\times\left(\sum_{k=1}^n |\widehat{B}_{k+1}(u)|\frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}\right) \lesssim \left(\frac{1-|B(u)|}{1-|u|}\right)^2.$$

□

Using Lemma 5.1, we first obtain the Arazy–Fisher–Peetre inequality for $p=1$:

$$\|B\|_{B_1} \lesssim \int_{\mathbb{D}}|B''(u)|\mathrm{d}A(u) \lesssim \sum_{j=1}^n\int_{\mathbb{D}}\frac{1-|z_j|^2}{|1-\overline{z_j}u|^3}\mathrm{d}A(u) + I_{2,0}(B) \lesssim n.$$

We used Dyn'kin's inequality (2.6) and the fact that, by the well-known estimates (see [HKZ, Theorem 1.7]), each integral in the above sum does not exceed some absolute constant, which does not depend on z_j. Finally, for $1<p<\infty$, we have

$$\begin{aligned}\|B\|_{B_p}^{\star\,p} &\asymp \int_{\mathbb{D}}|B''(u)|^p(1-|u|^2)^{2p-2}\mathrm{d}A(u)\\ &\le \big(\sup_{u\in\mathbb{D}}|B''(u)|(1-|u|^2)^2\big)^{p-1}\int_{\mathbb{D}}|B''(u)|\mathrm{d}A(u) \lesssim n,\end{aligned}$$

since $\sup_{u\in\mathbb{D}}|f''(u)|(1-|u|)^2 \le 2\|f\|_{H^\infty}$.

Proof of Theorem 2.3. Let $f\in\mathcal{R}_n^+$; there exists $\sigma\in\mathbb{D}^n$ such that $f\in K_{\widetilde{B}_\sigma}$, $\widetilde{B}_\sigma = zB_\sigma$. Then, by Proposition 4.2 we have

$$\|f'\|_{A_p(\alpha)} \le \|f\|_{BMOA}\left(I_{p,\alpha}(\widetilde{B}_\sigma)\right)^{\frac{1}{p}}$$

for any $\alpha>-1$ and $1<p<\infty$. Now applying Theorem 2.4, we obtain $\|f\|^\star_{A_p^1(\alpha)} \le K_{p,\alpha}\|f\|_{BMOA}\|B'_\sigma\|_{A_p(\alpha)}$. Finally, note that

$$\|\widetilde{B}'_\sigma\|_{A_p(\alpha)} \le \|zB'_\sigma\|_{A_p(\alpha)} + \|B_\sigma\|_{A_p(\alpha)} \lesssim \|B'_\sigma\|_{A_p(\alpha)}.$$

□

Remark. In Subsection 2.2.2, we have shown how to deduce Peller's inequality (2.1) from Theorem 2.3 and the result of Arazy–Fischer–Peetre (2.5). Let us show that for $p\ge 2$ one can give a very simple proof which uses only Proposition 4.2 and Dyn'kin's estimate $I_{2,0}(\widetilde{B}_\sigma)\le 8(n+2)$, where $n=\deg B_\sigma$. Indeed, in this case, we have

$$\begin{aligned}I_{p,\,p-2}(\widetilde{B}_\sigma) &= \int_{\mathbb{D}}(1-|u|^2)^{p-2}\left(\frac{1-|\widetilde{B}_\sigma(u)|^2}{1-|u|^2}\right)^{p-2+2}\mathrm{d}A(u)\\ &= \int_{\mathbb{D}}\left(1-|\widetilde{B}_\sigma(u)|^2\right)^{p-2}\left(\frac{1-|\widetilde{B}_\sigma(u)|^2}{1-|u|^2}\right)^2\mathrm{d}A(u) \le I_{2,0}(\widetilde{B}_\sigma).\end{aligned}$$

It remains to apply Proposition 4.2 with $\alpha=p-2$.

6. An elementary proof of Peller's inequality for $p > \frac{1}{2}$

In this section we prove the inequality

$$\|f\|_{B_p} \le cn^{\frac{1}{p}} \|f\|_{BMOA} \tag{6.1}$$

for $1 \ge p > \frac{1}{2}$ using the integral representations of the derivatives in model spaces. It is well known and easy to see that, for $p > \frac{1}{2}$,

$$\|f\|_{B_p}^p \asymp |f(0)|^p + |f'(0)|^p + |f''(0)|^p + \int_{\mathbb{D}} |f'''(u)|^p (1-|u|^2)^{3p-2} \mathrm{d}A(u).$$

Thus in what follows it is the last integral (which we denote $\|f\|_{B_p}^{\star\star}$) that we will estimate.

Let Θ be an inner function and let $f \in K_\Theta$. Then, by Lemma 4.1, $|f'''(u)| = \left|\left\langle f, z^3 (k_u^\Theta)^4 \right\rangle\right| \le \|k_u^\Theta\|_4^4 \|f\|_{BMOA}$, and so

$$\|f\|_{B_p}^{\star\star} \le \|f\|_{BMOA}^p \int_{\mathbb{D}} \|k_u^\Theta\|_4^{4p} (1-|u|^2)^{3p-2} \mathrm{d}A(u).$$

Lemma 6.1. *For any $u \in \mathbb{D}$,*

$$\|k_u^\Theta\|_4^4 = \frac{(1+|u|^2)(1-|\Theta(u)|^4)}{(1-|u|^2)^3} - \frac{4\mathrm{Re}\,(u\Theta'(u)\overline{\Theta(u)})}{(1-|u|^2)^2}.$$

Proof. The lemma follows from straightforward computations based on the formula $f'(u) = \langle f, z(1-\overline{u}z)^{-2}\rangle$. We omit the details. $\square$

We continue to estimate $\|f\|_{B_p}^{\star\star}$. Since $1+|u|^2 \le 2$ and $1-|\Theta(u)|^4 \le 2(1-|\Theta(u)|^2)$, we have

$$\|k_u^\Theta\|_4^4 \le \frac{4(1-|\Theta(u)|^2)}{(1-|u|^2)^3} - \frac{4\mathrm{Re}\,(u\Theta'(u)\overline{\Theta(u)})}{(1-|u|^2)^2}.$$

From now on assume that Θ is a finite Blaschke product $B = \prod_{k=1}^n b_{z_k}$. Then

$$\begin{aligned} uB'(u)\overline{B(u)} &= u|B(u)|^2 \frac{B'(u)}{B(u)} = u|B(u)|^2 \sum_{k=1}^n \left(\frac{1}{u-z_k} + \frac{\overline{z_k}}{1-\overline{z_k}u} \right) \\ &= |B(u)|^2 \sum_{k=1}^n \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2} + |B(u)|^2 \sum_{k=1}^n \frac{z_k(1-|z_k|^2)(1-|u|^2)}{|1-\overline{z_k}u|^2(u-z_k)}. \end{aligned}$$

Denote the last term by $S_1(u)$. Since $|B(u)| \le |b_{z_k}(u)|$, we have

$$|S_1(u)| \le \sum_{k=1}^n \frac{(1-|u|^2)(1-|z_k|^2)}{|1-\overline{z_k}u|^3},$$

whence (recall that $p \le 1$)

$$\int_{\mathbb{D}} \frac{|S_1(u)|^p}{(1-|u|^2)^p} (1-|u|^2)^{3p-2} \mathrm{d}A(u) \le \sum_{k=1}^n \int_{\mathbb{D}} \frac{(1-|z_k|^2)^p}{|1-\overline{z_k}u|^{3p}} (1-|u|^2)^{3p-2} \mathrm{d}A(u) \lesssim n,$$

since, by [HKZ, Theorem 1.7], each integral in the above sum does not exceed some constant depending only on p, but not on z_k.

Thus, to prove (6.1), it remains to estimate the weighted area integral of the difference

$$S_2(u) = \frac{1-|B(u)|^2}{(1-|u|^2)^2} - \frac{|B(u)|^2}{1-|u|^2}\sum_{k=1}^n \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}.$$

We use again the notations $B_k = \prod_{l=1}^{k-1} b_{z_l}$ (assuming $B_1 \equiv 1$) and $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$. It is easy to see that

$$\frac{1-|B(u)|^2}{1-|u|^2} = \sum_{k=1}^n |B_k(u)|^2 \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}. \tag{6.2}$$

Hence,

$$\begin{aligned} S_2(u) &= \sum_{k=1}^n |B_k(u)|^2 \cdot \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2} \cdot \frac{1-|\widehat{B}_k(u)|^2}{1-|u|^2} \\ &= \sum_{k=1}^n \sum_{l=k}^n \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2} \cdot \frac{1-|z_l|^2}{|1-\overline{z_l}u|^2} \cdot |B_l(u)|^2. \end{aligned}$$

Note that, by a formula analogous to (6.2), but without squares,

$$\frac{1-|B(u)|}{1-|u|} = \sum_{k=1}^n |B_k(u)| \frac{1-|b_{z_k}(u)|}{1-|u|} \geq \frac{1}{2}\sum_{k=1}^n |B_k(u)| \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2}.$$

Hence,

$$4\left(\frac{1-|B(u)|}{1-|u|}\right)^2 = \sum_{k=1}^n \sum_{l=1}^n |B_k(u)| \cdot |B_l(u)| \cdot \frac{1-|z_k|^2}{|1-\overline{z_k}u|^2} \cdot \frac{1-|z_l|^2}{|1-\overline{z_l}u|^2}.$$

Denote the last double sum by $S_3(u)$. Since $|B_kB_l| \geq |B_l|^2$, $l \geq k$, we see that $S_2(u) \leq S_3(u)$. Now we have

$$\begin{aligned} \int_{\mathbb{D}} \frac{|S_2(u)|^p}{(1-|u|^2)^p}(1-|u|^2)^{3p-2}\mathrm{d}A(u) &\leq 4^p \int_{\mathbb{D}} \left(\frac{1-|B(u)|}{1-|u|}\right)^{2p} (1-|u|^2)^{2p-2}\mathrm{d}A(u) \\ &\lesssim I_{2p,2p-2}(B) \lesssim \|B'\|^{2p}_{B_{2p}} \lesssim n. \end{aligned}$$

Here we used Theorem 2.4 to estimate $I_{2p,2p-2}(B)$ (recall that $2p > 1$) and the Arazy–Fisher–Peetre inequality (2.5). □

7. Radial-weighted Bergman norms of the derivative of finite Blaschke products

Again, let $n \geq 1$, $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ and let B_σ be the finite Blaschke product corresponding to σ. For any $1 < p < \infty$ and $\alpha > -1$, we set

$$\varphi_n(p, \alpha) = \sup\left\{ \|B'_\sigma\|_{A_p(\alpha)} : \sigma \in \mathbb{D}^n \right\}.$$

Note that for any $n \geq 1$, $\varphi_n(p, \alpha) = \varphi_1(p, \alpha) = \infty$ if and only if $\alpha < p-2$. Indeed, if $\alpha \geq p-2$, then $\varphi_n(p, \alpha) \leq \varphi_n(p, p-2) \asymp n^{\frac{1}{p}}$ by the Arazy–Fisher–Peetre inequality (2.5). For $\alpha < p-2$, consider the test function b_r, $r \in (0,1)$. It is easily seen (see, e.g., [HKZ, Theorem 1.7]) that $\|b'_r\|_{A_p(\alpha)} \to \infty$ as $r \to 1-$.

We have seen in Subsection 2.2.2 how the estimate $\varphi_n(p,\,p-2) \asymp n^{\frac{1}{p}}$ implies Peller's inequality (2.1). It could be of interest to find a more general estimate (for other values of α and p) of $\varphi_n(p,\,\alpha)$. Notice that for each fixed p, the function $\alpha \mapsto \varphi_n(p,\,\alpha)$ is decreasing and there exists the second critical value $\alpha_p \geq -1$,

$$\alpha_p = \inf\left\{\alpha > -1 :\ \sup_n \varphi_n(p,\,\alpha) < \infty\right\}.$$

The sequence $\{\varphi_n(p,\,\alpha)\}_{n\geq 1}$ may be unbounded and, thus, have a nontrivial asymptotics if and only if $p-2 \leq \alpha \leq \alpha_p$. In this notation we can rewrite Theorem 2.3 as

$$\|f\|^{\star}_{A^1_p(\alpha)} \lesssim \varphi_n(p,\,\alpha)\,\|f\|_{BMOA}\,,$$

for any $f \in \mathcal{R}_n^+$, $1 < p < \infty$, $p-2 \leq \alpha \leq \alpha_p$.

We will show now that $\alpha_p = p-1$, and so $p-1$ is the second critical value of α, as is expected from Theorem 2.2.

Proposition 7.1. *For any $p > 1$, $\alpha_p = p-1$.*

Proof. By the Schwarz–Pick lemma, we have that for any $\sigma \in \mathbb{D}^n$,

$$\|B'_\sigma\|^p_{A_p(\alpha)} \leq I_{p,\alpha}(B_\sigma),$$

and for any $\alpha > p-1$,

$$I_{p,\alpha}(B_\sigma) \leq \int_{\mathbb{D}} (1-|u|^2)^\alpha \left(\frac{1-|B_\sigma(u)|^2}{1-|u|^2}\right)^p \mathrm{d}A(u) \lesssim \int_{\mathbb{D}} \frac{1}{(1-|u|)^{p-\alpha}} \mathrm{d}A(u) < \infty,$$

and thus $\alpha_p \leq p-1$ for each p.

Next we show that $\alpha_p \geq p-1$. Let us consider the set $\sigma = (0,\,\ldots,\,0) \in \mathbb{D}^n$, for which $B_\sigma(z) = z^n$. In this case, we have

$$\|B'_\sigma\|^p_{A_p(\alpha)} = \left\|nz^{n-1}\right\|^p_{A_p(\alpha)} = n^p \int_0^1 r\,(1-r^2)^\alpha \int_{\mathbb{T}} |r\xi|^{p(n-1)}\,\mathrm{d}m(\xi)\,\mathrm{d}r,$$

which gives

$$\|B'_\sigma\|^p_{A_p(\alpha)} = n^p \int_0^1 (1-r^2)^\alpha r^{p(n-1)+1} \mathrm{d}r,$$

and

$$\beta(pn-p+2,\,\alpha+1) \leq \frac{\|B'_\sigma\|^p_{A_p(\alpha)}}{n^p} \leq 2^\alpha \beta(pn-p+2,\,\alpha+1),$$

where β stands for the *Beta function* $\beta(x,\,y) = \int_0^1 r^{x-1}(1-r)^{y-1}\mathrm{d}r$. Let $\alpha = p-1-\varepsilon$, $\varepsilon > 0$. Then by the standard Γ-function asymptotics, we obtain

$$\|B'_\sigma\|^p_{A_p(\alpha)} \geq \Gamma(\alpha+1) n^p \frac{\Gamma(pn-p+2)}{\Gamma(pn+\alpha-p+3)} \sim_{n\to\infty} \Gamma(\alpha+1) n^p (pn)^{\varepsilon-p},$$

whence $\sup_n \varphi_p(\alpha,\,n) = \infty$. $\square$

8. Remarks on Dolzhenko's inequalities

8.1. **Proof of Dolzhenko's inequalities for** $1 \le p \le 2$**.** In [Dol, Theorem 2.2] E.P. Dolzhenko proved that for any $f \in \mathcal{R}_n^+$,

$$\|f'\|_{A_p} \lesssim \begin{cases} n^{1-\frac{1}{p}} \|f\|_{H^\infty}, & 1 < p \le 2, \\ \log n \, \|f\|_{H^\infty}, & p = 1, \end{cases} \tag{8.1}$$

where the constants involved in $\lesssim$ may depend on p only. Let us show that these inequalities (and even with $BMOA$-norm in place of H^∞-norm) are direct corollaries of Proposition 4.2 and the following simple lemma.

Lemma 8.1. *For any Blaschke product B of degree n we have*

$$I_{p,0}(B) = \int_{\mathbb{D}} \left(\frac{1-|B(u)|^2}{1-|u|^2} \right)^p \mathrm{d}A(u) \lesssim \begin{cases} n^{p-1}, & 1 < p \le 2, \\ \log n, & p = 1. \end{cases} \tag{8.2}$$

Proof. Clearly, the integral over the disc $\left\{|z| \le 1 - \frac{1}{n}\right\}$ has the required estimate. The estimate over the annulus $\left\{1-\frac{1}{n} \le |z| < 1\right\}$ follows from the result of Dyn'kin ($I_{2,0}(B) \lesssim n$) and the Hölder inequality. Indeed, for $1 \le p < 2$,

$$\int_{\left\{1-\frac{1}{n} \le |z| < 1\right\}} \left(\frac{1-|B(u)|^2}{1-|u|^2} \right)^p \mathrm{d}A(u) \le (I_{2,0}(B))^{\frac{p}{2}} \left(\pi(1 - \left(1 - \frac{1}{n}\right)^2 \right)^{1-\frac{p}{2}} \lesssim n^{p-1}.$$

$\square$

Now inequality (8.1) follows from (8.2) and from the inequality $\|f'\|_{A_p(\alpha)} \le \|f\|_{BMOA} (I_{p,\alpha}(B_\sigma))^{\frac{1}{p}}$ which holds for any function $f \in K_{B_\sigma}$ (see Proposition 4.2). It should be mentioned, however, that Dolzhenko proves his inequalities for more general domains than the unit disc.

8.2. **An extension of Dolzhenko's inequalities to the range** $p > 2$**.** The case $p > 2$ is also treated by Dolzhenko (see the last inequality in [Dol, Theorem 2.2]), but the corresponding analog is of somewhat different nature. As the example $f(z) = (1 - rz)^{-1}$ with $r \to 1-$ shows, there exist no estimate of $\|f'\|_{A^p}$ in terms of $\|f\|_{BMOA}$ and $n = \deg f$.

Here we obtain another extension of Dolzhenko's result for $p > 2$.

Theorem 8.2. *Let $2 < p \le \infty$, let $f \in \mathcal{R}_n^+$, $n \ge 1$, and let $1/\overline{\lambda}_1, \dots, 1/\overline{\lambda}_n$ be its poles (repeated according to multiplicities). We have*

$$\|f'\|_{A_p} \lesssim n^{\frac{1}{p}} \left(\sum_{k=1}^{n} \frac{1+|\lambda_k|}{1-|\lambda_k|} \right)^{1-\frac{2}{p}} \|f\|_{BMOA}. \tag{8.3}$$

Moreover, the inequality (8.3) is asymptotically sharp in the following sense: for any $r \in (0, 1)$ there exists $g \in \mathcal{R}_n^+$ having $\frac{1}{r}$ as a pole of multiplicity n such that

$$\|g'\|_{A_p} \gtrsim n^{1-\frac{1}{p}} \left(\frac{1}{1-r} \right)^{1-\frac{2}{p}} \|g\|_{BMOA}. \tag{8.4}$$

Proof. We first prove (8.3). Set $\sigma = (\lambda_1, \dots, \lambda_n)$, so that $f \in K_{\widetilde{B}_\sigma}$. By Proposition 4.2 and Dyn'kin's inequality (2.6),

$$\|f'\|_{A_p}^p \le \|f\|_{BMOA}^p \int_{\mathbb{D}} \left(\frac{1-|\widetilde{B}_\sigma(u)|^2}{1-|u|^2} \right)^p \mathrm{d}A(u) \lesssim n \, \|f\|_{BMOA} \sup_{u \in \mathbb{D}} \left(\frac{1-|\widetilde{B}_\sigma(u)|^2}{1-|u|^2} \right)^{p-2}.$$

By (6.2), we have

$$\frac{1-|\widetilde{B}_\sigma(u)|^2}{1-|u|^2} \le 1 + \sum_{k=1}^n \frac{1-|\lambda_k|^2}{|1-\overline{\lambda_k}u|^2} \lesssim \sum_{k=1}^n \frac{1+|\lambda_k|}{1-|\lambda_k|}.$$

This proves (8.3). Now we prove (8.4). Take $g = b_{-r}^n(u)$, $r \in (0,1)$, then

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} \left|b'_{-r}(u)\right|^2 \left|b'_{-r}(u)\right|^{p-2} |b_{-r}(u)|^{p(n-1)} \, \mathrm{d}A(u).$$

Taking $v = b_{-r}(u)$ as the new variable and using the fact that $u = b_{-r}(v)$, we get

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} \left|b'_{-r}(b_{-r}(v))\right|^{p-2} |v|^{p(n-1)} \, \mathrm{d}A(v).$$

Since $b'_{-r} \circ b_{-r}(v) = -\frac{(1+rv)^2}{1-r^2}$, we obtain

$$\|g'\|_{A_p}^p = \frac{n^p}{(1-r^2)^{p-2}} \int_{\mathbb{D}} |1+rv|^{2(p-2)} \, |v|^{p(n-1)} \, \mathrm{d}A(v).$$

Supposing that $p \ge 2$, we have

$$\int_{\mathbb{D}} |1+rv|^{2(p-2)} \, |v|^{p(n-1)} \, \mathrm{d}A(v) \asymp \int_{\mathbb{D}} |v|^{p(n-1)} \, \mathrm{d}A(v) = \frac{2}{pn-p+2},$$

whence

$$\|g'\|_{A_p}^p \asymp \frac{n^{p-1}}{(1-r)^{p-2}}.$$

Since $\|g\|_{BMOA} = 1$ this completes the proof (8.4). □

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